Journal of Statistical Physics, Vol. 21, No. 1, 1979

Simple Kinetic Model of Symmetry Breaking

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Received December 6, 1977; revised July 5, 1978 and January 2, 1979

We consider a dynamical system described by a set of random variables $N_i(t)$ and depending on a parameter R controlling its stability. If $R < R_c$ the system is stable and the N_i have some symmetry properties in the statistical sense (i.e., with respect to time averaging). If $R > R_c$ the system is unstable and the nonlinear dynamics of the N_i may lead to an asymptotic stationary state which does not possess the symmetries of the stable system. We show that the dynamics of symmetry breaking resembles a phase transition in the limit of many variables.

KEY WORDS: Benard convection; Volterra equations; symmetry breaking; nonequilibrium fluctuations.

1. INTRODUCTION

Many systems in physics and nonphysical sciences show the phenomenon of symmetry breaking: e.g., the appearance of organized patterns in nonequilibrium fluids (Benard and Taylor cells, structures produced by oscillatory strains in a viscoelastic fluid[1,3,6]), and the laserlike phase transition predicted at thermal equilibrium in a many-body system described by the Dick Hamiltonian.[9] These various systems usually depend on some external parameter λ such that the dynamics exhibits a bifurcation at a critical value $\lambda = \lambda_c$. The equilibrium state at $\lambda < \lambda_c$ possesses some symmetry property in the statistical sense (or is invariant under a definite group of transformations), this property being lost at $\lambda > \lambda_c$. However, the kinetics of the symmetry breaking is essentially unknown, and we propose to investigate it on a simple and solvable model: a set of p ordinary differential equations of the Lotka Volterra type. A salient feature we want to show is that in the limit of large p (i.e., of many degrees of freedom) the system keeps its initial symmetry during

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0022-4715/79/0700-0033$03.00/0

a very long time (compared to the inverse of the instability rate), staying quasistationary. Then, the kinetics changes abruptly, leading to the final asymmetric equilibrium.

2. THE MODEL

We consider a system described by p variables $N_i(t)$ evolving in time according equations of the form

$$\dot{N}_i = N_i\left[\gamma_i - \sum_{j=1}^{p} \nu_{ij}N_j\right] + s_i \tag{1}$$

where γ_i and ν_{ij} are positive coefficients and the s_i are positive fluctuating source terms.

The linear growth rates γ_i depend on external constraints and may vary in time. When the γ_i are negative and constant the amplitudes of the N_i are damped in the absence of source terms. These are able to establish a statistical equilibrium. The symmetry property we shall assign to our system is that this statistical equilibrium is invariant with respect to any permutation among the N_i. This will be achieved by imposing the following conditions:

(a) The γ_i are i independent:

$$\gamma_i = \gamma \tag{2a}$$

(b)

$$\nu_{ij} = \nu(|i - j|) \tag{2b}$$

(c) The s_i themselves possess the symmetry property in the statistical sense:

$$\langle s_i \rangle = s \tag{3a}$$

$$\langle s_i s_j \rangle = \text{function of } |i - j| \tag{3b}$$

$$N_{i+p} = N_i \tag{4}$$

Now, we assume that, in a real experiment, the external parameters cause γ to vary in time in the fashion shown on Fig. 1a. If Δt is small enough ($\Delta t\, \gamma \ll 1$) the dynamics will be correctly approximated on long time scales by considering a function $\gamma(t)$ whose graph is a step function (see Fig. 1b):

$$\gamma(t) = \begin{cases} \gamma^-, & t < 0 \\ \gamma, & t > 0 \end{cases} \tag{5}$$

Before the onset of instability ($t < 0$) the amplitudes of the N_i are damped (in the absence of source terms), and any particular realization $\{N_i\}$ is short-lived. The statistical equilibrium is essentially determined by the dynamics of

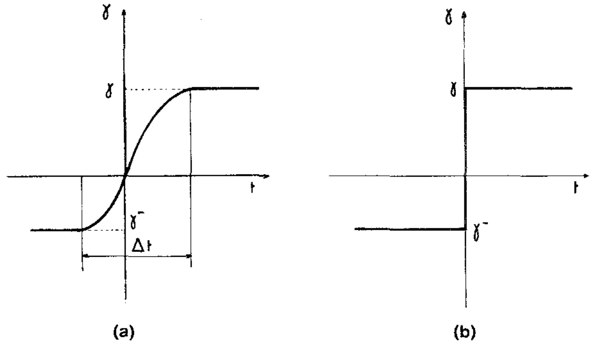

Fig. 1. Shape of $\gamma(t)$.

the s_i, which ensures that at any time the symmetry property is preserved. At $t = 0$ a particular realization $\{\hat{N}_i^0\}$ (among all possible fluctuating configurations $\{N_i^0\}$) is present, which is not invariant with respect to the permutations of N_i (the symmetry property is only a statistical one). The amplitudes of the N_i^0 are no longer damped, and they grow for $t > 0$. The deterministic evolution would lead the system over some characteristic time t_R toward an asymptotic stationary state with broken symmetry. We shall show in Section 4 that if the amplitudes of the s_i are small enough, and if other specific conditions are satisfied, the system remains weakly fluctuating around the deterministic solution (starting from N_i^0 at time zero). Therefore, we first study the evolution of the deterministic model.

3. THE DETERMINISTIC EVOLUTION

The equation of motion of the deterministic model are

$$\dot{N}_i = \gamma N_i - N_i \sum_j \nu_{ij} N_j \tag{6a}$$

$$\dot{N}_i(t = 0) = N_i^0 \tag{6b}$$

Equations (6a) and (6b) are of the Volterra type, and we have studied their properties in a preceding paper[(2)] from a more general point of view (arbitrary ν_{ij} and i-dependent growth rates). The particular features of our model (which must be fulfilled in view of the study of the dynamics of symmetry breaking) are (i) the growth rates are all equal, $\gamma_i = \gamma$; and (ii) the ν_{ij} are symmetric $\nu_{ji} = \nu_{ij}$.

It is known that property (ii) implies the existence of a Liapunov function $H = \sum_i \gamma N_i - \frac{1}{2} \sum_{ij} \nu_{ij} N_i N_j$, whose time derivative is positive definite at any time.[(7)] It can be shown[(3)] that this fact rules out any kind of cyclic or ergodic behavior of the solution curve of Eqs. (6a) and (6b) in the space of the N_i, and the trajectory tends asymptotically toward an equilibrium point. There exists a set of equilibrium points, which is obtained by setting the right-hand sides of Eqs. (6a) and (6b) equal to zero. It is defined by

$$\bar{N}_i = 0 \qquad \text{for} \quad i \in \{q\}$$

$\{q\}$ being an arbitrary subset of the p modes,

$$\sum_j \nu_{ij} \bar{N}_j = \gamma \qquad \text{for} \quad i, j \in \{p - q\}$$

the various equilibria being relevant only if their components $\bar{N}_i$ are all positive [it is indeed easily seen that, starting from positive initial conditions $N_i(t_0)$, the trajectory remains in the positive half-space of the N_i (see Ref. 2)]. Now, we have shown[2] that the condition

$$\nu_{ij} > \nu_{ii} \qquad \forall i, j \tag{7}$$

implies that all the above equilibria are unstable, except the one-mode equilibria. An equivalent statement of this property has been given by Haken[5] in the case of a simplified form of the ν_{ij}. Moreover, the existence of the above-mentioned H function ensures that this asymptotic state is actually attained whatever be the set of initial conditions.

An additional obvious remark on the structure of Eqs. (6a) and (6b) is that a flat initial distribution [$N_i(t_0)$ independent of i] remains flat during the evolution (this is due to the symmetry properties of the system). Therefore a solution with broken symmetry (i.e., a one-mode solution) appears because the random set of initial conditions is itself never exactly symmetric (with respect to the permutation of i indices) at the microscopic level.

Since the essential morphological properties of Eqs. (6a) and (6b) are properties (i) and (ii) supplemented by conditions $\nu_{ij} > \nu_{ii}$ ($j \neq i$), we have been led to study a simpler model obeying the above constraints, but whose mathematical analysis is easier. This model is defined by considering constant coefficients of interaction except for the terms ν_{ii}. Therefore, we write

$$\nu_{ij} = \nu_0[(1 + \beta) - \beta\, \delta_{ij}] \tag{8}$$

Therefore, ν_{ij} has the shape represented in Fig. 2. Then, taking γ^{-i} as unit time and making the change $N_i \to \nu_0^{-1} N_i$ [$s_i \to s_i = (\nu_0/\gamma)s_i$], we can write Eq. (6) as

$$\dot{N}_i = \left[1 - (\beta + 1) \sum_{j=1}^{p} N_j + \beta N_i\right] N_i \tag{9}$$

Introducing new variables $U_i = \int_0^t N_i(t')\, dt'$, we can formally integrate Eq. (9) as

$$N_i = \dot{U}_i = N_i^0 \exp(\beta U_i) \exp\left[t - \sum_{k=1}^{p} (1 + \beta) U_k\right] \tag{10}$$

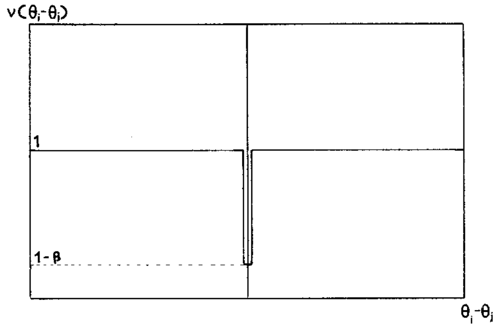

Fig. 2. The angular dependence of interaction kernel $\nu(|\theta_i - \theta_j|)$ when $\beta_{ij} = C^{te}$.

from which we obtain that

$$\frac{dU_i}{N_i^0} e^{-\beta U_i} = \frac{dU_j}{N_j^0} e^{-\beta U_j} = \cdots = \beta^{-1}\dot{g}(t) \tag{11}$$

where $g(t)$ is function of time, but not depending on index i. Integration of Eqs. (11) with initial conditions $U_i(0) = 0$ and $g(0) = 0$ gives

$$g(t) = \frac{1 - e^{-\beta U_i}}{N_i^0} = \frac{1 - e^{-\beta U_j}}{N_j^0} = \cdots \tag{12}$$

from which

$$N_i(t) = \beta^{-1} N_i^0 \dot{g}(t)/[1 - N_i^0 g(t)] \tag{13}$$

Let us remark that a consequence of Eq. (13) is that if several occupation numbers N_i, N_j,... are initially equal, they remain equal at any time.

The existence of function $g(t)$ permits us to reduce the kinetics of the set of the N_i to the kinetics of one variable. Indeed, we can eliminate the U_i with the help of Eqs. (12) and (13). Then Eq. (10) may be rewritten in terms of $g(t)$ as

$$\dot{g} = \beta e^t \prod_{i=1}^{p} (1 - N_i^0 g)^\mu \tag{14}$$

with $\mu = (1 + \beta)/\beta$. Equation (14) may be integrated over time, giving

$$e^t - 1 = \int_0^g \frac{dg'}{\prod_i (1 - N_i^0 g')^\mu} \tag{15}$$

and the kinetic problem is then reduced to a study of the function $e^t = F(g)$. However, tractable expressions for the integral in Eq. (15) do not exist and it is easier to work on the differential equation (14).

Since β is positive and $g(0) = 0$, we see from Eq. (14) that $\dot{g}(0) > 0$ and

$\dot{g}(t)$ remains positive until $g(t)$ reaches the value $(N_\lambda{}^0)^{-1}$, with $N_\lambda{}^0 = \sup\{N_i{}^0\}$. It is easily seen that g cannot reach this value at finite time. The asymptotic evolution is obtained by replacing g by $(N_\lambda{}^0)^{-1}$ in each factor entering Eq. (14) except in $(1 - N_\lambda{}^0 g)$. We obtain in this way

$$\dot{g} \underset{t\to\infty}{\approx} \beta e^t \left\{ \prod_{i\neq\lambda} \left(1 - \frac{N_i{}^0}{N_\lambda{}^0}\right)^\mu \right\} (1 - N_\lambda{}^0 g)^\mu \tag{16}$$

Putting $\epsilon_\lambda = 1 - N_\lambda{}^0 g$, the integration of Eq. (16) yields

$$\epsilon_\lambda^{-1/\beta} \sim N_\lambda{}^0 \prod_{i\neq\lambda} \left(1 - \frac{N_i{}^0}{N_\lambda{}^0}\right)^\mu e^t$$

Using Eq. (13), we conclude that $N_{i\neq\lambda} \to 0$, $N_\lambda \to 1$ when $t \to \infty$. Therefore, we verify the general result, which was stated above in the case of arbitrary (symmetric) ν_{ij} that the asymptotic evolution leads to the survival of only one species.

In order to obtain more information on the overall evolution we shall first consider the simple case where the initial distribution is flat $[N_i(t_0) = n_0]$. Equation (14) thus reduces to

$$\dot{g} = \beta e^t (1 - n_0 g)^{p\mu} \tag{17}$$

and its solution is

$$1 - n_0 g = [1 + (p\mu - 1)\beta n_0 (e^t - 1)]^{-1/(p\mu - 1)} \tag{18}$$

from which

$$N_i(t) = N(t) = \frac{N_0 e^t}{1 + (p\mu - 1)\beta n_0 (e^t - 1)} \tag{19}$$

A sketch of the curve $g(t)$ is given in Fig. 3, where one sees that, after a linear stage of exponential growth $[g(t) \approx \beta(e^t - 1)]$, during a time interval

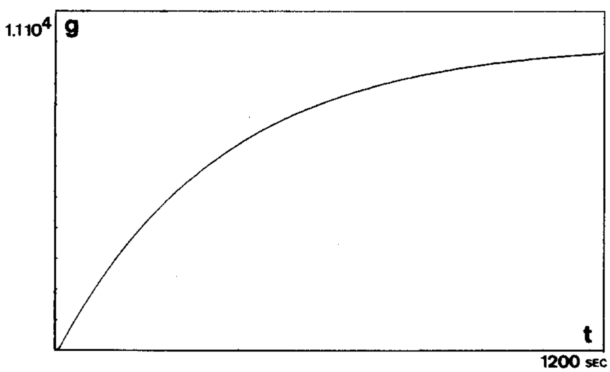

Fig. 3. The evolution of $g(t)$ in the case of uniform initial spectrum $N_i(0) = 10^{-3}$, $p = 100$.

$t_{NL} \sim \log(1/p\mu n_0)$ we enter the nonlinear regime where the growth is much slower, $1 - n_0 g(p\mu\beta n_0 e^t)^{-1/p\mu}$ (for large p). The evolution of $N(t)$ is represented in Fig. 4, the two curves corresponding respectively to the case where the initial value n_0 is (a) smaller or (b) greater than the saturation value $1/\beta(p\mu - 1)$ [we shall concentrate on case (a) in the following].

We want now to study the kinetics of many modes (p large), with arbitrary initial distributions. It is helpful to consider the particular case where the initial spectrum is uniform, except for the λth mode, whose occupation number is larger ($N_{i\neq\lambda} = n^0$, $N_\lambda{}^0 = n_\lambda$, $n_\lambda > n_0$). The corresponding equation for g is

$$\dot{g} = \beta e^t(1 - n_0 g)^{p\mu}(1 - n_\lambda g)^\mu \tag{20}$$

The form of Eq. (20) suggests, in the limit of large p, that the kinetics is controlled in a first stage by the factor $(1 - n_0 g)^{p\mu}$, as long as $(1 - n_\lambda g)^\mu$ is not too small. Later we enter the asymptotic stage, during which $g(t)$ obeys the asymptotic equation (16) in the case $N_{i\neq\lambda} = n_0$. We can give an evaluation of the first stage by considering the logarithmic derivative of Eq. (20), namely

$$\frac{\ddot{g}}{\dot{g}} = 1 - \mu\dot{g}\left(\frac{n_\lambda}{1 - n_\lambda g} + \frac{pn_0}{1 - n_0 g}\right) \tag{21}$$

The first stage will be defined as the time interval during which the $n_{0\lambda}$ term is small compared to the n_0 term in Eq. (21), and an order of magnitude of the duration of this stage is obtained by writing

$$\frac{n_\lambda}{1 - n_\lambda g(t_R)} \sim \frac{pn_0}{1 - n_0 g(t_R)} \tag{22a}$$

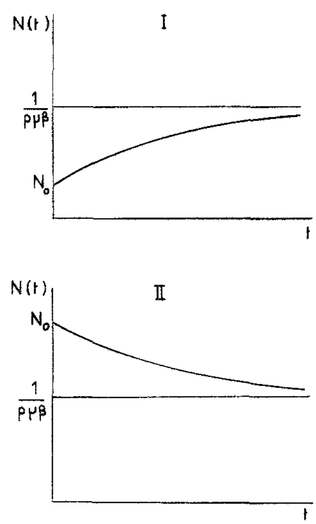

Fig. 4. The two kinds of evolution of $N_i(t)$ in the case of uniform initial spectrum: (top) $N_i(0) < 1/p\mu\beta$; (bottom) $N_i(0) > 1/p\mu\beta$.

and by assuming that $g(t_R)$ is still given by expression (18), i.e., according to the "homogeneous kinetics." Then, t_R is given by

$$p\mu\beta n_0 e^{t_R} \sim \left(1 - \frac{n_0}{n_\lambda}\right)^{-p\mu} - 1 \sim \left(1 - \frac{n_0}{n_\lambda}\right)^{-(p\mu)} \tag{22b}$$

(for large p).

In the two limiting cases $n_0/n_\lambda = 1 - \eta$ and $n_0/n_\lambda = \eta$ [where $(p\mu)^{-1} \ll \ll \eta \ll 1$], one obtains, respectively,

$$n_0/n_\lambda = 1 - \eta, \qquad t_R \sim p\mu \log(1/\eta) \tag{22c}$$

$$n_0/n_\lambda = \eta, \qquad t_R \sim p\mu\eta \tag{22d}$$

The two stages of the kinetics of $g(t)$ are pictured in Fig. 5.

In the limit of large p, $g(t)$ first grows exponentially, then for $t > t_{NL}$ we have a quasistationary stage where the evolution is not qualitatively different from that defined by solution (18). During the same time interval $N_0(t)$ and $N_\lambda(t)$ grow monotonically, very soon reaching the homogeneous saturation level $(p\mu\beta)^{-1}$, while $N_\lambda(t)$ is still slowly growing [according to expression (13) and taking account of the growth of g for $t < t_R$]. For time t of the order of t_R the $(1 - n_\lambda g)$ term becomes important in Eq. (20); the $g(t)$ curve departs from the homogeneous curve and tends to approach its asymptotic limit $g(\infty) = 1/n_\lambda$ (see Fig. 4). After a transient time Δt the kinetics obeys the asymptotic equation (16). We want to show, through a short (but rough) argument, that $\Delta t/t_R \to 0$ in the limit of large p. A majoration of Δt may be obtained in the following way. Let us put $\epsilon_\lambda = 1 - n_\lambda g(t_R)$. We shall evaluate the time $t_r + \Delta t$ at which ϵ appreciably deviates from its initial value (say

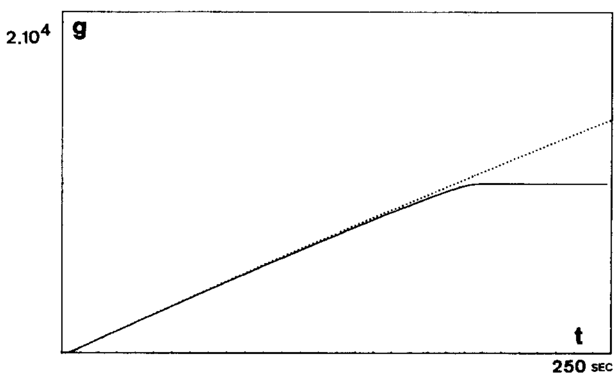

Fig. 5. Evolution of $g(t)$ in the case of 1000 equal initial values [$N_0(0) = 10^{-5}$] and one different one [$N_1(0) = 10^{-4}$]; the dotted curve represents the evolution of g in the case of uniform initial spectrum [$N_0(0) = 10^{-5}$].

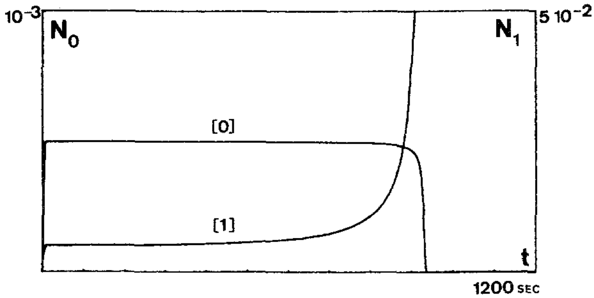

Fig. 6. The evolution of $N_0(t)$ and $N_1(t)$ for $N_0(0)/N_1(t) = 1 - \eta$, $p = 100$, $\eta = 10^{-2}$.

$\epsilon_\lambda \to \epsilon_\lambda' = \epsilon_\lambda/10$) when one assumes that $\epsilon(t)$ evolves according to the slowest kinetics, namely the asymptotic one. We obtain, using Eq. (16),

$$e^{\Delta t} - 1 = \frac{(\epsilon_\lambda')^{-1/\beta} - (\epsilon_\lambda)^{-1/\beta}}{n_\lambda}\left(1 - \frac{n_0}{n_\lambda}\right)^{-p\mu} e^{-t_R}$$

According to Eq. (22b), $e^{-t_R}(1 - n_0/n_\lambda)^{-p\mu} \sim p\mu\beta n_0$, and we have

$$e^{\Delta t} - 1 \approx e^{\Delta t} \approx (p\mu\beta n_0)\frac{(\epsilon_\lambda')^{-1/\beta} - (\epsilon_\lambda')^{-1/\beta}}{n_\lambda} \tag{23}$$

with $\epsilon_\lambda \approx (1/p)(n_\lambda/n_0 - 1)$.

Therefore, we conclude that $\Delta t < \log p$ for large p. Using the above evaluation for t_R, $\Delta t/t_R < (\log p)/p$ for $p \to \infty$. Since the decay rate of $N_0(t)$ in the asymptotic stage is of the order of unity [cf. Eq. (16)], we conclude that the selection of the λ mode in the vicinity of $t = t_R$ takes a vanishingly small time (compared to the overall time t_R characterizing the kinetics). This behavior recalls a phase transition. Numerical calculations on Eq. (20) confirm the above conclusions. We show in Figs. 6 and 7 the evolution of $N_0(t)$ and $N_\lambda(t)$ in the two limiting cases $n_0/n_\lambda = 1 - \eta$ and $n_0/n_\lambda = \eta$.

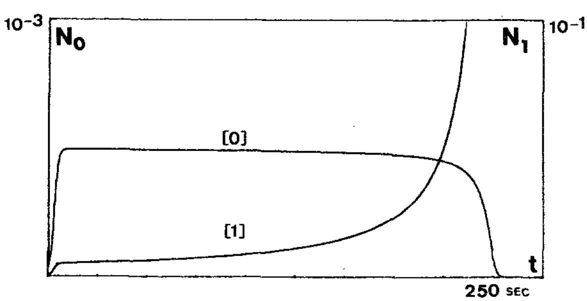

Fig. 7. The evolution of $N_0(t)$ and $N_1(t)$ for $N_0(0)/N_1(0) = \eta$, $p = 1000$, $\eta = 10^{-1}$.

Let us finally consider the general case where the initial spectrum $n_i = N_i(0)$ is characterized by its average value $\overline{N_i(0)} = n_0$ $[n_i(0) = n_0 + \delta n_i]$ and its mean square deviation $\sigma^2 = (1/p) \sum (\delta n_i/n_0)^2$ [later we shall consider an average over a statistical ensemble of independent random variables $N_i(0)$] and by the data of $n_\lambda = \sup\{N_i(0)\}$. Assuming a relatively small dispersion, we obtain the following expression for $\ddot{g}/\dot{g}$ [after using a first-order expansion of the $(1 - N_i{}^0 g)$ factors $(i \neq \lambda)$ in terms of $\delta n_i g/(1 - n_0 g)$]:

$$\frac{\ddot{g}}{\dot{g}} = 1 - \mu\dot{g}\left\{\frac{1}{1 - n_0 g}\left[pn_0 + \frac{g}{(1 - n_0 g)^2}\left(\sum_i \delta n_i{}^2\right)\right] + \frac{n_\lambda}{1 - n_\lambda g}\right\} \tag{24}$$

We see that the evolution is quasi-"homogeneous" if $p\sigma^2 n_0{}^2 g/(1 - n_0 g)^2 < pn_0$. If $n_\lambda \ll n_0$ (which will usually be the case for statistically independent $N_i{}^0$), this condition reduces to $n_0 g < 1/\sigma^2$. Taking account of a dispersion of initial values $N_i(0)$ around n_0 will noticeably modify the kinetics only if $n_0 g$ reaches the value $1/\sigma^2$ before the value $1/n_\lambda$, that is, if $1/n_0\sigma^2 < 1/n_\lambda$.

Finally, we say a few words on the more general problem where the interaction kernel ν_{ij} actually depends on $|i - j|$.

We shall consider, as an example, the case where the ν_{ij} take the form

$$\nu_{ij} = 1 + \beta_{ij} - \beta_{ii} - \delta_{ij}$$

where

$$\beta_{ij} = \frac{(5 + \cos\theta_{ij})^2(1 - \cos\theta_{ij})^2}{(5 + \cos\theta_{ij})^3 - (27/4)(1 + \cos\theta_{ij})} + \frac{(5 - \cos\theta_{ij})^2(1 + \cos\theta_{ij})^2}{(5 + \cos\theta_{ij})^3 - (27/4)(1 - \cos\theta_{ij})} \tag{25}$$

where $\theta_{ij} = \theta_i - \theta_j$. Here θ_i is a characteristic angle associated with ith populations. This example comes from the hydrodynamics of Benard convection, and the above ν_{ij} are the interaction coefficients between roll structures along directions θ_i and θ_j.[8] We comment on this question in our conclusion.

A simple analysis of the kinetics is no longer available, and the detailed form of the function $\nu(|i - j|)$ plays a role. The main features of the kinetics remain the same as in the above simplified model, and, in particular, numerical integration shows (see Fig. 8) the tendency, for large p, to a three-step evolution: (i) the linear growth of the initial spectrum, (ii) the quasisaturation of the N_i during a time roughly proportional to p, (iii) a sudden catastrophe where all the modes but one disappear.

However, a new feature is now possible: the selected mode is not always the one whose initial occupation number is maximum. Taking account of the monotonic decrease of $\nu(|\theta_i - \theta_j|)$ (excepting the hole at the origin), we may

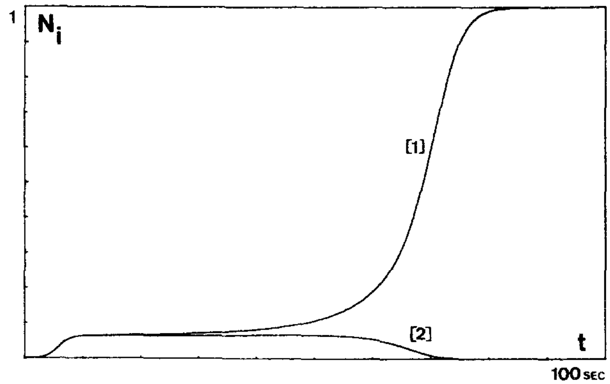

Fig. 8. Evolution of 20 species obeying Eq. (12) with the true Newell–Whitehead kernel (only two species evolutions are represented). The initial values are $N_i(0) = \exp[-(i-1)^2/1000] \times 10^{-4}$.

expect that, starting from an initial distribution with many modes around some value θ_0, plus one mode at θ_1 well separated from θ_0 with smaller occupation number N_1 (see Fig. 9), we obtain the selection of N_1 through nonlinear competition. Indeed, the growth of the isolated mode is less impeded by its interactions with the other modes than is the growth of any mode inside the main packet with stronger mode–mode interactions inside the packet [$\nu(|\theta_i - \theta_j|)$ at small angles is large]. Numerical integration of Eq. (9) confirms the existence of this phenomenon (see Fig. 10).

We add the following general comment. The discontinuity of the Newell–Whitehead interaction kernel ν_{ij} at $i = j$ results in $\nu_{ii} < \nu_{ij}\ \forall(i, j)$ and therefore ensures that only one-mode equilibria are stable. One can study a continuous version of Eqs. (6a) and (6b), namely an integral equation of the form

$$\partial_p N(\theta, t) = N(\theta, t)\left\{\int \alpha(\theta - \theta')N(\theta', t)\, d\theta'\right\}$$

for suitable analytic forms of the kernel $\alpha(|\theta|)$. It can be shown that if $\alpha(|\theta|)$ is a monotonic decreasing function of θ, one obtains, starting from an arbitrary initial angular distribution $N(\theta, 0)$, an asymptotic isotropization of the distribution.

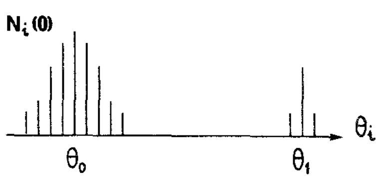

Fig. 9. Qualitative shape of an initial spectrum leading to inversion phenomena.

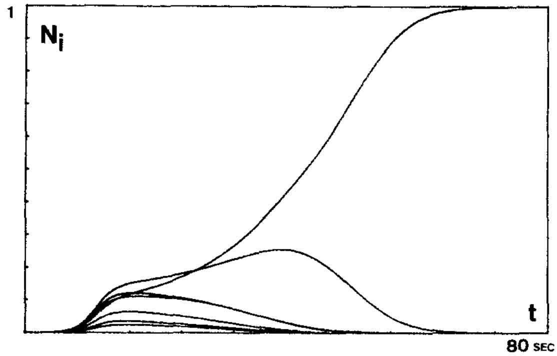

Fig. 10. Numerical simulation of the inversion phenomenon. The initial values of the seven modes are 10^{-4}, 9.5×10^{-5}, 8.5×10^{-5}, 3×10^{-5}, 7×10^{-5}, 2×10^{-5}, 5×10^{-5}.

4. THE EFFECT OF THE FLUCTUATIONS

4.1. General Considerations

In the preceding section, we have described the evolution of inhomogeneous initial conditions, i.e., of an asymmetric initial state of the system (with respect to permutations of the N_i). We showed the difficult merging of an initially larger population competing with many others, but we have not really pictured a symmetry-breaking process. We could find the analog of the phase transition with broken symmetry of a physical system with many degrees of freedom (such as the superconductivity transition). In order to do that we must restore the fluctuations. Indeed, it is clear that in the thermal system the initial symmetry property is of a statistical nature (it does not exist at the microscopic, fluctuating level). At the time $t = 0$ when the physical parameter controlling the bifurcation reaches its critical value, the symmetric state is destabilized, while there appears a new stable, asymmetric state. However, the initial symmetry of the system demands that the symmetric solution still exists (but is unstable). Therefore, the establishment of the final state with broken symmetry may result from a competition between the destabilized unstable and a disymmetric solution, grown from a disymmetric fluctuating state which was present at $t = 0$. Such a phenomenon may be described easily in our simple model by introducing fluctuations. These have two effects:

1. They introduce a statistical distribution of the $N_i(0)$ at initial critical time. Given the probability law of the $N_i(0)$, we can evaluate the relative weight of any particular realization of the $\{N_i(0)\}$ set in the final state.

2. They act in the kinetic equations through the presence of fluctuating source terms, which are eventually weakly perturbed by the growth of the instability (for instance, in the case of the coupling of the system with a larger thermal bath). Such terms tend to restore the initial symmetry of the system.

Of course these two effects are usually not independent, but they may act on very different time scales, especially if the characteristic time of the instability is small enough.

We shall first consider the effect of initial fluctuations by giving a reasonable probability law of these fluctuations: we assume the $N_i(0) = n_i$ to be statistically independent random variables distributed around an average value n_0 according a Boltzmann-like probability law:

$$f(n_i/n_0) = e^{-n_i/n_0} \tag{26}$$

Therefore, the probability law of the largest population n_λ is easily obtained and is given by

$$\varpi(X) = p(1 - e^{-X})^{p-1}e^{-X} \qquad (X = n_\lambda/n_0) \tag{27}$$

p being the number of species. In the limit of large p, $\varpi(X)$ is sharply peaked around $\bar{X} = \log p$. Now it results from the remarks at the end of the preceding section that small enough fluctuations of the n_i around n_0 will negligibly modify the deterministic, two-component kinetics $\{n_0, n_\lambda\}$. The condition of small fluctuations ($\sigma^2 < \bar{n}_\lambda/n_0$) is obviously satisfied with the above probability law. Therefore, the time of catastrophe t_R associated with a particular realization of the set of n_i is still given by Eq. (22b), which gives in the limit of large p, $t_R \sim p(n_\lambda/n_0)^{-1}$. Averaging over initial n_i and using the probability law (27), we find

$$\bar{t}_R \sim p/\log p \tag{28}$$

We now want to account for the effect of fluctuating source terms in the kinetic equations and show that, under definite conditions on the magnitude of these terms, the final state of the system will be only slightly fluctuating around the deterministic state.

4.2. Qualitative Evaluation of the Effect of the Fluctuating Source Terms

Let us first come back to the physical picture of the evolution when the linear growth rate is steadily increased from negative values to positive ones (Fig. 2). As long as $\gamma < 0$ ($t < 0$) we expect an adiabatic evolution toward a nontrivial equilibrium state slowly changing in time. This will happen if the variation of $\gamma(t)$ is sufficiently low (say, if $\gamma^{-1}\, d\gamma/dt$ is much smaller than the correlation time of the s_i) and if the s_i and γ are related through the usual fluctuation dissipation theorem. When $t > 0$ the instability starts, and the fluctuating source terms may drive, on the long run, our system far from the deterministic one-population solution. Remember also that, neglecting the possible effect of the instability on source terms, these terms tend to restore the initial symmetry.

In order to study qualitatively the effect of the positive fluctuating source term s in Eq. (1), we use the former simplified form of the interaction coefficients

$$\nu_{ij} = (1 + \beta) - \beta\,\delta_{ij}$$

Therefore, we have to deal with

$$\dot{N}_i = N_i - N_i\left[(1 + \beta)\sum_j N_j - \beta N_i\right] + s_i, \qquad s_i = \nu_0 s_i/\gamma \tag{29a}$$

$$N_i(0) = N_i^0 \tag{29b}$$

Let us now set

$$N_i(t) = \bar{N}_i(t) + \delta N_i(t) \tag{30}$$

where $\bar{N}_i(t)$ are the above solutions of the deterministic model starting from the initial conditions (29b). The $\delta N_i(t)$ obeys the following equations:

$$\begin{aligned}\delta\dot{N}_i = s_i + \delta N_i - \delta N_i\left[(1 + \beta)\sum_{j=1}^{p} \bar{N}_j - 2\beta\bar{N}_i\right]\\ - \bar{N}_i\left[\sum_{j=1}^{p} (1 + \beta)\,\delta N_j\right] - \delta N_i\left[(1 + \beta)\sum_{j=1}^{p} \delta N_j - \beta\,\delta N_i\right]\end{aligned} \tag{31a}$$

$$\delta N_i(0) = 0 \tag{31b}$$

In the following we shall suppose, and verify later, that under convenient conditions the nonlinear term $\delta N_i[(1 + \beta)\sum_{j=1}^{p} \delta N_j - \beta\,\delta N_i]$ in Eq. (31a) plays a negligible role in the kinetics. Therefore, the evolution equations for the δN_i reduce to

$$\delta\dot{N}_i \simeq s_i + \delta N_i - \delta N_i\left[(1 + \beta)\sum_{j=1}^{p} \bar{N}_j - 2\beta\bar{N}_i\right] - \bar{N}_i\sum_{j=1}^{p}(1 + \beta)\,\delta N_j$$

$$\delta N_i(0) = 0$$

Now in Section 3, we have shown that for large p the evolution of $\bar{N}_i$ proceeds in three steps:

(a) *An exponential growth* for $0 < t < t_{\mathrm{NL}} \simeq \log(1/p\mu n_0)$, at the end of which $\bar{N}_i$ are of the order of $1/p\mu\beta$.

(b) *A quasistationary stage* for $t_{\text{NL}} < t < t_R$.

(c) *A final stage* for $t > t_R$, where all the N_i but one suddenly vanish.

We shall give an estimation of the order of magnitude of $\delta N_i(t)$ at the end of these three steps.

It is easily seen that for $t < t_{\text{NL}}$ the δN_i have an exponential growth and that they behave for $t \sim t_{\text{NL}}$ like

$$\delta N_i \sim \int_0^{t_{\text{NL}}} e^{\tau} s_i(t_{\text{NL}} - \tau)\, d\tau$$

If s is a characteristic value of the random positive functions s_i, we obtain an order of magnitude

$$\delta N_i(t_{\text{NL}}) \sim e^{t_{\text{NL}}} s \sim s/p\mu n_0 \tag{32}$$

It is also easily seen (taking into account the above renormalizations) that the order of magnitude of n_0 is

$$n_0 \sim s\gamma^-/\gamma$$

(See Eq. (5) for the definitions of γ and γ^-.)

As a consequence, the δN_i will be small at the end of the linear stage if

$$\gamma/\gamma^- p\mu \ll 1/p\mu\beta \Rightarrow \gamma \ll \gamma^-/\beta$$

Now let us consider the quasistationary evolution, where the N_i are all of the order of $(p\mu\beta)^{-1}$.

Equation (31a) reads

$$\begin{aligned} \delta \dot{N}_i &\simeq s_i + \delta N_i - \delta N_i \left[1 - \frac{2\beta}{p\mu\beta}\right] \frac{1}{p} \sum \delta N_j \\ &\simeq s_i + \delta N_i \frac{2\beta}{p(1+\beta)} - \frac{1}{p} \sum \delta N_j \end{aligned} \tag{33}$$

from which we obtain

$$\sum \delta \dot{N}_i \simeq \sum s_i - \sum \delta N_j \left\{1 - \frac{2\beta}{p(1+\beta)}\right\} \tag{34}$$

For large enough p the solutions of Eq. (34) always lead to a stationary limit

$$\left\langle \sum \delta N_j \right\rangle \sim \left\langle \sum s_i \right\rangle \tag{35}$$

It is also easily verified that, after time T_R, the δN_i will be of the order of

$$\delta N_i(T_R) \sim \frac{1}{p\mu} \frac{\gamma}{\gamma^-} \exp\left[\frac{2\beta}{p(1+\beta)} T_R\right] \tag{36}$$

With T_R of the order of $p\mu$, we obtain

$$\delta N_i(T_R) \sim \frac{1}{p\mu}\frac{\gamma}{\gamma^-}\exp\left[\frac{2\beta}{p(1+\beta)}p\mu\right] \sim \frac{1}{p\mu}\frac{\gamma}{\gamma^-}\exp(+2)$$

Again, if $\gamma/\gamma^- \ll 1$, the δN_i will remain small during the quasistatic evolution.

Now let us consider what happens when the saturation is attained. We have in this stage

$$N_i = 0, \quad i \neq k; \qquad N_k = 1$$

from which we deduce

$$\delta \dot{N}_i = s_i + \delta N_i - \delta N_i[(1+\beta)] = s_i - \beta\,\delta N_i \qquad (i \neq k)$$

$$\begin{aligned}\delta \dot{N}_k &= s_k + \delta N_k - \delta N_k[(1+\beta) - 2\beta] - \left[(1+\beta)\sum \delta N_j\right] \\ &= s_k + \beta\,\delta N_k - (1+\beta)\sum \delta N_j \\ &= s_k - \delta N_k - \sum_{j\neq k} \delta N_j\end{aligned}$$

It is easily seen that for $i \neq k$ the δN_i tend to be such that

$$\delta N_i(t) = \int_0^\infty e^{-\beta\tau} s_i(t-\tau)\,d\tau$$

from which

$$\sum_{j\neq k} \delta N_j = \int_0^\infty e^{-\beta\tau} \sum_{j\neq k} s_j(t-\tau)\,d\tau$$

with T_R of the order of $p\mu$, we obtain

$$\delta N_i(T_R) \sim \frac{1}{p\mu}\frac{\gamma}{\gamma^-}\exp\left[\frac{2\beta}{p(1+\beta)}p\mu\right] \sim \frac{1}{p\mu}\frac{\gamma}{\gamma^-}\exp(2) \tag{37}$$

Again, if $\gamma/\gamma^- \ll s$ the ∂N_i will remain small during the quasistatic evolution.

Now, let us consider what happens when saturation is attained. We have in this stage

$$N_i = 0, \quad i \neq k; \qquad N_k = 1 \tag{38}$$

from which we obtain

$$\delta \dot{N}_i = s_i - \beta\,\delta N_i \qquad (i \neq k) \tag{39a}$$

$$\delta \dot{N}_k = s_k - \delta N_k - \sum_{j\neq k} \delta N_j \tag{39b}$$

For $i \neq k$ the δN_i behave for large t like

$$\delta N_i(t) \simeq \int_0^\infty e^{-\beta\tau} s_i(t - \tau)\, d\tau \tag{40a}$$

and

$$\sum_{j \neq k} \delta N_j(t) = \int_0^\infty e^{-\beta\tau} \sum_{j \neq k} s_j(t - \tau)\, d\tau \tag{40b}$$

Inserting the above value of $\sum_{j \neq k} \delta N_j$ in Eq. (39b), we obtain for large t

$$\delta N_k(t) = \int_0^\infty \left[s_k(t - \tau) - \int_0^\infty e^{-\beta\tau'} \sum_{j \neq k} s_j(t - \tau - \tau')\, d\tau' \right] e^{-\tau}\, d\tau \tag{41}$$

In order of magnitude, we obtain

$$\delta N_i \sim s/\beta \quad (i \neq k); \qquad \delta N_k \sim -\left(\frac{p - 1}{\beta} - 1 \right) s \tag{42}$$

Obviously, the above evaluations are valid if

$$ps/\beta \ll 1 \tag{43a}$$

$$\gamma/\gamma^- \ll 1 \tag{43b}$$

[in order to justify the neglect of the nonlinear terms $\delta N_i\ \delta N_j$ in Eqs. (31a) and (31b)].

Condition (43a) has the meaning that, in the equilibrium state, the population of the dominant (or macroscopic) species must be much larger than the amplitudes of the other fluctuating species.

5. CONCLUDING REMARKS

We have given some insight into the kinetics of symmetry breaking in a simple dynamical system. Our model is certainly quite elementary, but it points out important features, such as the long relaxation time and the sudden transition in the limit of many variables. In the course of this paper we have alluded to the problem of Benard convection. We think that it is an example where these considerations find an interesting application, and suggest experimental investigations. Of course, intrinsic difficulties must be solved: essentially the problem of reducing the continuous hydrodynamic field to a set of discrete hydrodynamic modes (whose intensities would be our N_i), and also of dealing with the slow diffusion of these modes in configuration space. This study will be the purpose of a subsequent paper.

REFERENCES

1. F. H. Busse, *Fluid Mech.* **30**:625 (1967).
2. J. Coste, J. Peyraud, P. Coullet, and A. Chenciner, *Theor. Pop. Biol.* **14**:165 (1978).
3. S. H. Davis, *J. Fluid Mech.* **30**:465 (1967).
4. R. Graham and H. Pleiner, *Phys. Fluids* **18**:130 (1974).
5. H. Haken, *Rev. Mod. Phys.* **47**:67 (1975).
6. R. Krishnamurti, *J. Fluid Mech.* **33**:445, 457 (1968).
7. R. H. McArthur, *Proc. Nat. Acad. Sci.* **64**:1369 (1969).
8. A. C. Newell and J. A. Whitehead, *J. Fluid Mech.* **38**:279 (1969).
9. G. Vallee, Thèse, Université de Nice (1977).
10. J. Swift and P. C. Hohenberg, *Phys. Rev.* **18**:319 (1977).